Systems of Equation

A Formal Study

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SYSTEMS OF EQUATIONS

Introduction

In this math assignment we consider systems of equations. Roughly speaking, a system of equations is just a collection of equations with a common set of unknowns. In solving such systems, we try to find values for the unknowns that simultaneously satisfy each of the equations in the system. In Mini-Lesson 1.0 we review two procedures that are often taught in beginning algebra for solving systems involving two linear equations in two unknowns. An important technique for solving larger systems of equations is developed in Mini-Lessons 2.0 and 3.0; this technique is known as ***Substitution*** *and* ***addition-subtraction*** *methods.*

• How to use the mini-lesson and other reference materials:

a. When you are reading the mini-lesson and you come upon definitions of a mathematical term or a major concept, make sure to write them in your toolbox-book.

b. Read the explanatory material in the mini-lesson slowly and carefully to acquire an in-depth understanding of the material (concepts, procedure, and strategies)[1].

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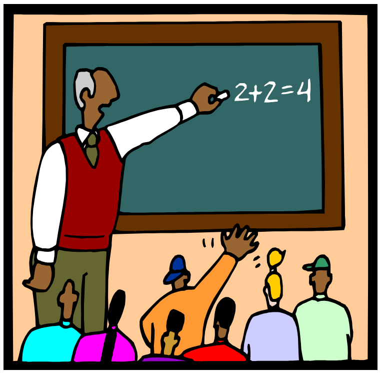
[1] Refer to **your Student Handbook for Mathematics**

Mini- Lesson 1.0



Both in theory and in applications, it's often necessary to solve two equations in two unknowns. You have been introduced to the idea of simultaneous equations in a previous math class; however, to put matters on a firm foundation, we begin our current study with the basic definitions. When we speak of a **linear equation in two variables** we mean an equation of the form.

$ax + by = c$

where the constants *a* and *b* are not both zero. The two variables needn't always be represented by the letters *x and y*, of course; it is the form of the equation that matters.

An ordered pair of numbers (x_0, y_0) is said to be a **solution of the linear equation** $ax + by = c$ if we can find a true statement when we replace *x* and *y* in the equation by x_0 and y_0, respectively. For example, the ordered pair (3, 2) is a solution of the equation x – y = 1, since 3 – 2= 1. On the other hand, (2, 3) is not a solution of x – y = 1, since 2 – 3 ≠ 1.

Now consider a **system** of two linear equations in two unknowns:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

If we can find an ordered pair that is a solution to both equations, then we say that ordered pair is a **solution of the system.** Sometimes, to emphasize the fact that a solution must satisfy both equations, we refer to the system as a pair of **simultaneous equations**. A system that has at least one solution is said to be **consistent.** If there are no solutions, the system is **inconsistent**.

Example 1 Consider the system

$$\begin{cases} x + y = 2 \\ 2x - 3y = 9 \end{cases}$$

(a) Is (1, 1) a solution of the system?

(b) Is (3, -1) a solution of the system?

Solution

(a) Although (1, 1) is a solution of the first equation, it is not a solution of the system because it does not satisfy the second equation. (Check this out for yourself.)

(b) (3, -1) satisfies the first equation:

$$3 + (-1) = 2 \quad \text{True}$$

(3, -1) satisfies the second equation:

$$2(3) - 3(-1) = 9 \quad \text{True}$$

Since (3, -1) satisfies both equations, it is a solution of the system.

We can observe an important perspective on systems of linear equations by looking at Example 1 visually on the Cartesian Plane.

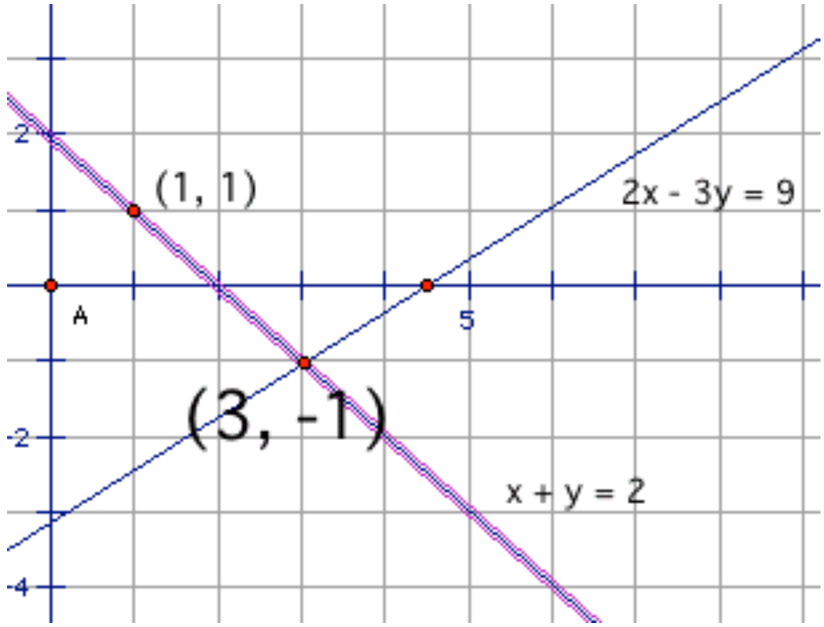


Table 1 shows how each of the statements in that example can be rephrased using the geometric ideas with which we now know.

Algebraic Idea	Corresponding Geometric Idea
1. The ordered pair (1, 1) is a solution of the equation x + y = 2.	1. The point (1, 1) lies on the line x + y = 2. See Figure above.
2. The ordered pair (1, 1) is not a solution of the equation 2x – 3y = 9.	2. The point (1, 1) does not lie on the line 2x – 3y = 9. See Figure above.
3.The ordered pair (3, -1) is a solution of the system $\begin{cases} x+y=2 \\ 2x-3y=9 \end{cases}$	3. The point (3, -1) lies on both of the lines x + y = 2 and 2x – 3y = 9. See Figure above.

In Example 1, we checked to see whether, or not the ordered pair (3, -1) is a solution of the system

$$\begin{cases} x + y = 2 \\ 2x - 3y = 9 \end{cases}$$

Are there any other solutions of this particular system? No: Figure 1 above shows us that there are no other solutions, since (3, -1) is clearly the only point common to both lines. In the next section of this math assignment we'll look at two important methods for solving systems of linear equations in two unknowns. But even before we consider these methods, I want to summarize something about the solutions of linear systems.

Possibilities For Solutions of Linear Systems

Given a system of two linear equations in two unknowns, exactly one of the following cases must occur.
Case 1 The graphs of the two linear equations intersect in exactly one point. Thus, there is exactly one solution to the system. See Figure 2.
Case 2 The graphs of the two linear equations are parallel lines. Therefore the lines do not intersect, and the system has not solution. See Figure 3.
Case 3 The two equations actually represent the same line. Thus, there are infinitely many points of intersection and correspondingly infinitely many solutions. See Figure 4.

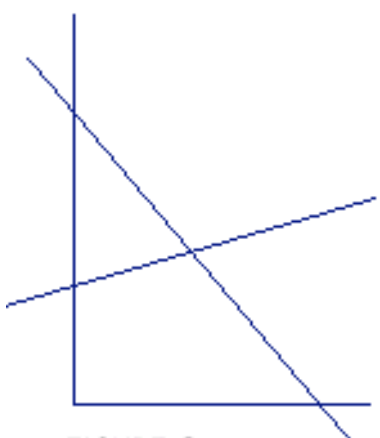


FIGURE 2
A consistent system with exactly one soluton

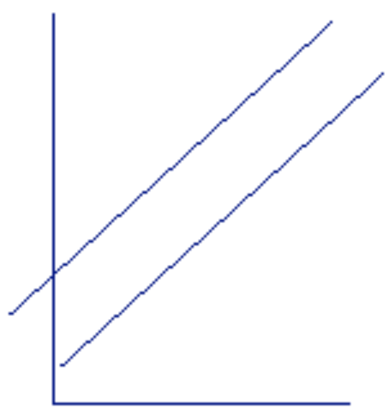


FIGURE 3
An inconsistent system has no soluton

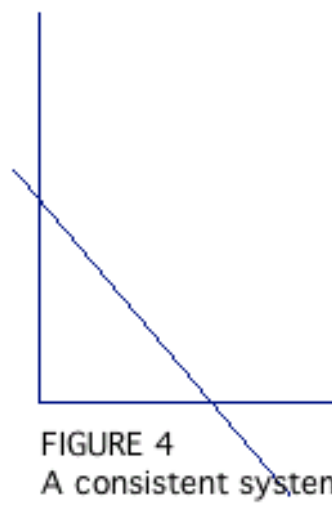
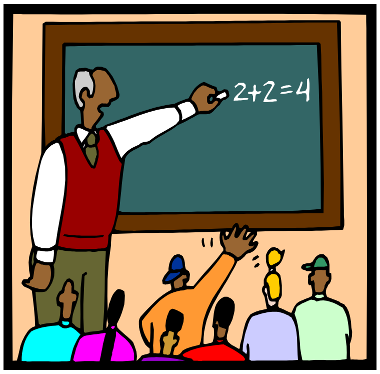


FIGURE 4
A consistent system with infinitely many solutons

Mini- Lesson 2.0



We are going to study two methods from basic algebra for solving systems of two linear equations in two unknowns. These methods are the **substitution method** and the **addition-subtraction method.** I will begin by demonstrating the substitution method. Consider the system

$$\begin{cases} 3x + 2y = 17 \quad \text{equation (1)} \\ 4x - 5y = -8 \quad \text{equation (2)} \end{cases}$$

We first choose one of the two equations and then use it to express one of the variables in terms of the other. In the case that we are studying, neither equation appears particularly simpler than the other, so let's just start with the first equation and solve for *x* in terms of in terms *y*. We have

$$3x = 17 - 2y$$

$$x = \frac{1}{3}(17 - 2y) \quad \text{equation 3}$$

Now we use equation (3) to substitute for *x* in the equation that we have not yet used, namely, equation (2). This give us

$$4\left[\frac{1}{3}(17 - 2y)\right] - 5y = -8$$

$$4(17 - 2y) - 15y = -24 \quad \text{multiplying by 3}$$

$$\text{-23y} = \text{-92}$$

$$\text{y} = 4$$

The value y = 4 that we have just obtained can now be used in equation (3) to find *x*. Replacing *y* with 4 in equation (3) yields

$$x = \frac{1}{3}[17 - 2(4)] = \frac{1}{3}(9) = 3$$

We have now found the $x = 3$ and $y = 4$. As you can easily check, this pair of values indeed satisfies both of the original equations. We write our solution as the ordered pair (3, 4). Figure 5 (below) summarizes this situation graphically. It shows that the system is consistent and that (3,4) is the only solution.

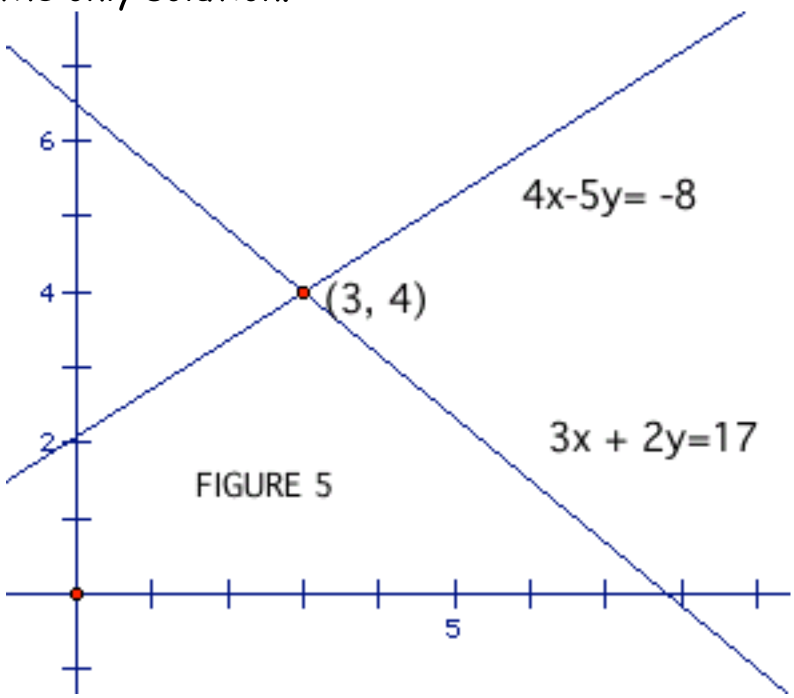


FIGURE 5

Generally speaking, it is not necessary to graph the equation in a given system in order to decide whether the system is consistent. Rather, this information will emerge as you attempt to follow the algebraic method of solution.

EXAMPLE 2 Solve the system

$$\begin{cases} \frac{3}{2}x - 3y = -9 \\ x - 2y = 5 \end{cases}$$

Solution We use the substitution method. Since it is easy to solve the second equation for *x*, we begin there;

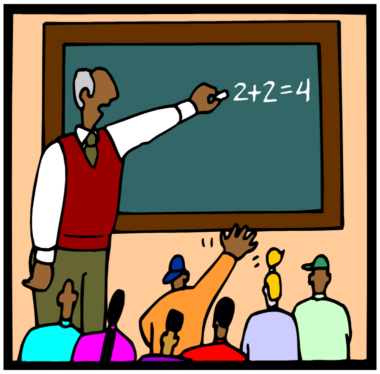
$$x - 2y = 4$$
$$x = 4 + 2y$$

Now we substitute this result in the first equation of our system to obtain

$$\frac{3}{2}(4 + 2y) - 3y = -9$$
$$6 + 3y - 3y = -9$$
$$6 = -9 \quad \text{False}$$

Since the substitution process leads us to this obviously false statement, we conclude that the given system has not solution; that is, the system is ***inconsistent***. What can you say about the graphs of the two given equations?

Mini- Lesson 3.0



Now let's turn to the ***addition-subtraction*** method of solving systems of equations. Let's use the following problem.

$$\begin{cases} 2x + 3y = 5 \\ 4x - 3y = -1 \end{cases}$$

Notice that if we add these two equations, the result is an equation involving only the unknown x:

$$6x = 4$$

$$x = \frac{4}{6} = \frac{2}{3}$$

There are now several ways in which we can find the corresponding value of *y*. As you can easily check, substituting the value *x = 2/3* in either of the original equations leads to the result *y = 11/9*.

Another way to find *y* is by multiplying both sides of the first equation by –2. (You'll see why in a moment.) We show the work this way:

$2x + 3y = 5$	Multiply by -2	$-4x - 6y = -10$
$4x - 3y = -1$	No change	$4x - 3y = -1$

Adding the last two equations then gives us

$$-9y = -11$$

$$y = 11/9$$

The required solution is therefore (2/3, 11/9).

In the previous example, we were able to find *x* directly by adding the two equations. ***As the next example will show, it may be necessary first to multiply both sides of each equation by an appropriate constant.***

EXAMPLE 4 Solve the system

$$\begin{cases} 5x - 3y = 4 \\ 2x + 4y = 1 \end{cases}$$

Solution To eliminate *x*, we could multiply the second equation by 5/2 and then subtract the resulting equation from the first equation. However, to avoid working with fractions, we proceed as follows.

$$5x - 3y = 4 \quad \xrightarrow{\text{Multiply by 2}} \quad 10x - 6y = 8 \quad (4)$$

$$2x + 4y = 1 \quad \xrightarrow{\text{Multiply by 5}} \quad 10x + 20y = 5 \quad (5)$$

Subtracting equation (5) from equation (4) then yields

$$-26y = 3$$

$$y = -3/26$$

To find *x*, we return to the original system and work in a similar manner:

$$5x - 3y = 4 \quad \xrightarrow{\text{Multiply by 4}} \quad 20x - 12y = 16$$

$$2x + 4y = 1 \quad \xrightarrow{\text{Multiply by 3}} \quad 6x + 12y = 3$$

Upon adding the last two equations, we obtain

$$26x = 19$$

$$x = 19/26$$

The solution of the given system of equation is therefore (19/26, -3/26).

EXAMPLE 5 Find the area of the triangular region bounded by the line y = 3x. y = -1/2x +7, **and the x-axis;** see Figure 6.

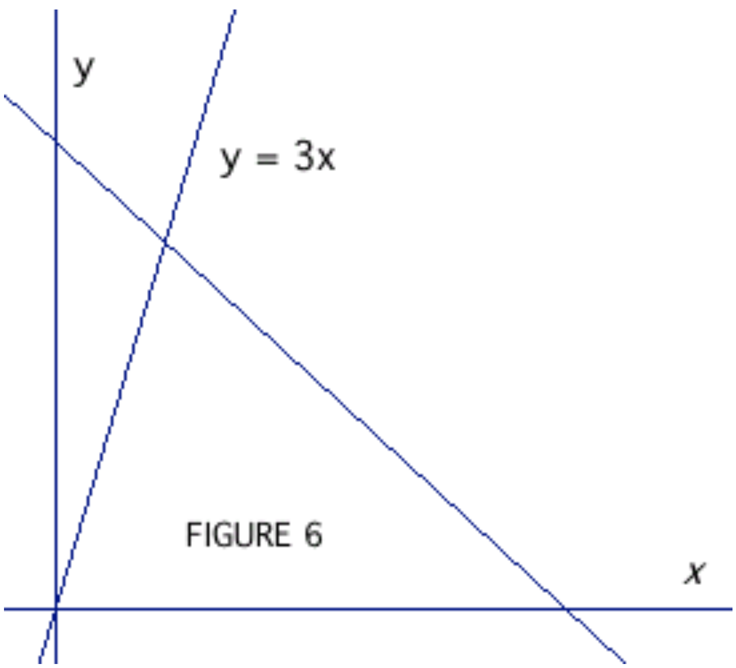


FIGURE 6

Solution To compute the area of the triangle, we need to know the base and the height. To determine the base, we first compute the x-intercept of the line $y = -\frac{1}{2}x + 7$:

$$0 = -\frac{1}{2}x + 7$$

$$\frac{1}{2}x = 7$$

$$x = 14$$

The height of the triangle is the y-coordinate of the intersection point of the lines y = 3x and y = -1/2x + 7 (see Figure 6). We can use the substitution method to solve this pair of simultaneous equations for *y*. From the equation y = 3x, we obtain *x* = 1/3*y*. Then, substituting this in the equation $y = -\frac{1}{2}x + 7$ yields

$$y = -\frac{1}{2}\left(\frac{1}{3}y\right) + 7$$

$$6y = -y + 42$$

$$7y = 42$$

$$y = 6$$

Now that we know that the base of the triangle is 14 units and the height is 6 units, we can compute the required area:

$$A = \frac{1}{2}bh = \frac{1}{2}(14)(6) = 42 \text{ square units}$$

EXERCISE SET

1. Which of the following are linear equations in two variables?
   a) 3x + 3y = 10 b) 2x + 4xy + 3y = 1
   c) u – v = 1 d) x = 2y + 6

2. Is (5, 1) a solution of the following system?

$$\begin{cases} 2x - 8y = 2 \\ 3x + 7y = 22 \end{cases}$$

3. Is (0, -4) a solution of the following system?

$$\begin{cases} \frac{1}{6}x + \frac{1}{2}y = -2 \\ \frac{2}{3}x + \frac{3}{4}y = 2 \end{cases}$$

4. Is (3, -2) a solution of the following system?

$$\begin{cases} \frac{2}{7}x - \frac{1}{5}y = \frac{44}{35} \\ \frac{1}{3}x - \frac{5}{4}y = \frac{7}{2} \end{cases}$$

In Exercises 5-9, use the substitution method to find all solutions of each systems.

5.
$$\begin{cases} 3x - 2y = -19 \\ x + 4y = -4 \end{cases}$$

6. $$\begin{cases} 4x + 2y = 3 \\ 10x + 4y = 1 \end{cases}$$

7. $$\begin{cases} 13x - 8y = -3 \\ -7x + 2y = 0 \end{cases}$$

8. $$\begin{cases} \dfrac{1}{3}x - \dfrac{2}{5}y = 4 \\ 7x + 3y = 27 \end{cases}$$

9. $$\begin{cases} x + y = \dfrac{5}{2} \\ 2x - 3y = 1 \end{cases}$$

In Exercises 10-12, use the addition-subtraction method to find all solutions of each system of equations.

10.
$$\begin{cases} 2x+3y=2 \\ 4x-3y=1 \end{cases}$$

11.
$$\begin{cases} 2x-3y=-2 \\ 2x+y=14 \end{cases}$$

12.
$$\begin{cases} 2x+3y=4 \\ 5x+6y=7 \end{cases}$$

I strongly suggest for Exercise 13-16 that you first clear both equations of fractions and decimals.

13.
$$\begin{cases} v=-4u+1 \\ u=\frac{1}{4}v-\frac{1}{4} \end{cases}$$

14. $$\begin{cases} \frac{1}{4}r + \frac{1}{3}s = \frac{5}{12} \\ \frac{1}{2}r + s = 1 \end{cases}$$

15. $$\begin{cases} 0.2x + 0.5y = 0.9 \\ 0.01x + 0.03y = -0.01 \end{cases}$$

16. $$\begin{cases} 2(x - y - 1) = 1 - 2x \\ 6(x - y) = 4 - 3(3y - x) \end{cases}$$

17. POINTS (-8, -16), (0, 10), and (12, 14) are three vertices of a parallogram. Find the coordinates of the fourth vertex if it is located in the three quadrant.

18. A line with equation ax+by = 3 passes through (6, 3) and (-1, -1). Find a and b without finding the slope.

19. Find the area of the triangular region in the first quadrant bounded by the x-axis and the lines y = 2x-5 and y = (-3/2)x + 3.