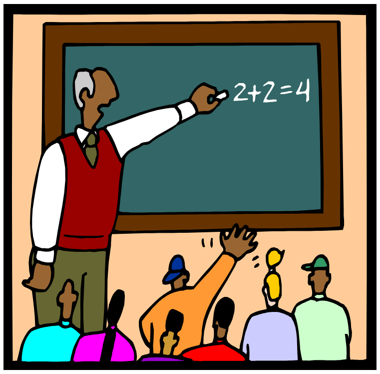


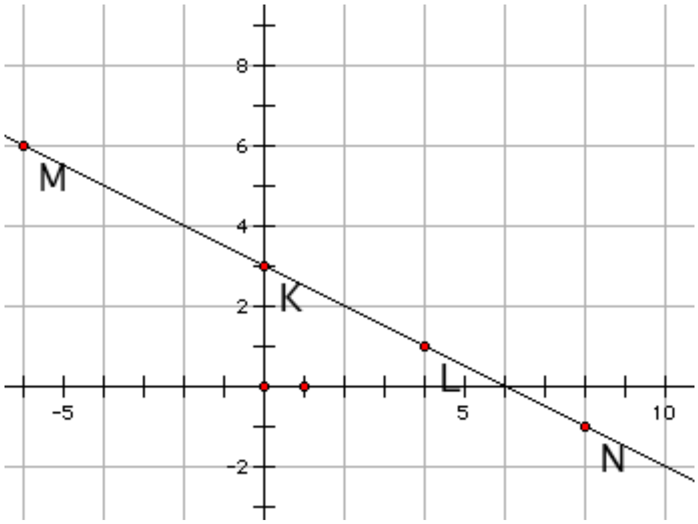
Mini- Lesson 1--Straight Lines in a Plane

You have recently worked with open sentences in a variety of ways. In each case we were concerned with open sentences in terms of one variable. Let us now consider cases with two variables. Such situations are common on mathematics, and we will investigate a few elementary types in some detail.

Suppose that we are interested in two variables, x and y, related so that $x + 2y = 6$. This is an open sentence in two variables, and we are interested in *ordered pairs* as replacements for the variables. The solution set consists of those ordered pairs of elements such that the first plus twice the second is 6. We could phrase our statement regarding the solution set as we did for other open sentences. That is, simply write $\{(x, y) \mid x + 2y = 6\}$. The solution set is this case contains an infinite number of elements from the set $\Re \times \Re$. We may write a <u>partial tabulation</u> as follows:

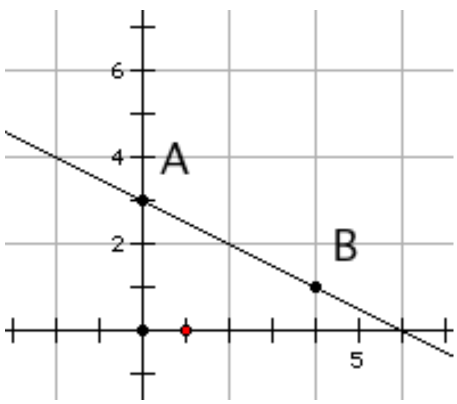
$$\{(x, y) \mid x + 2y = 6\} = \{(0, 3), (4, 1), (8, -1), (-6, 6), (5, \tfrac{1}{2}), (-3, \tfrac{9}{2}), (1.2, 2.4)\}$$

A practical procedure for obtaining ordered pairs in a situation such as this is to replace one of the variables by some real number and then solve the resulting equation for the remaining variable in terms of the constants. Thus, if $x = \frac{5}{2}$, we would write $\frac{5}{2} + 2y = 6$. Therefore, $2y = \frac{7}{2}$ and $y = \frac{7}{4}$. The ordered pair having $\frac{5}{2}$ as the first member is therefore $(\frac{5}{2}, \frac{7}{4})$. Now let us graph these ordered pairs in a rectangular coordinate system. In the display below we have established a set of axes and have plotted the points associated with the values tabulated in our solution set. It appears that all the points lie on the same straight line (i.e. are collinear).

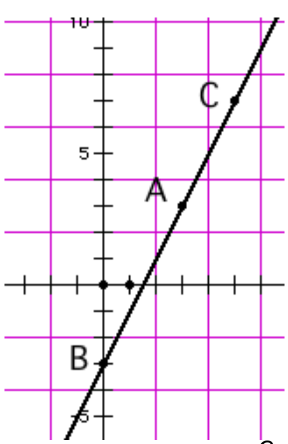


The speculation about the collinearity of the points in the preceding paragraph is indeed correct. We could proceed to prove that any equation of the form $ax + by + c = 0$ (such as $x + 2y = 6$) has a graph that is a straight line. However, I prefer simply to accept this assertion as true and then proceed to another pertinent matter.

Property 1.4 For a, b, c, **any real numbers, a and b not zero, the graphs of equations of the form** $ax + by + c = 0$ **are straight lines and conversely. We refer to** $ax + by + c$ **= 0 as a linear equation.**



With the added assurance given us by the property above, we could be much more confident in our approach to the set $\{(x, y) \mid x + 2y = 6\}$. We are sure that it is a linear equation. The graph of the solution set must be a straight line. Knowing this we could determine two ordered pairs, plot two points, and then simply draw the line determined by them. (In practice, we usually plot a third point as a check for our work.) We would then be sure that the graph of any ordered pair that satisfies the conditions $x + 2y = 6$ falls on the line and, further, that any point on the line has coordinates that upon substitution makes $x + 2y = 6$ a true statement. This graph is illustrated in the display below.



Example Graph $\{(x, y) \mid 2x - y = 3\}$

The ordered pairs (0, -3), (3, 3), and (5, 7) are in the truth set. The graph illustrated in the next display.

We are often interested in determined the points where the graph crosses the axes. The crossing point on the x-axis is referred to as the *x-intercept*. Since any point on the x-axis has a y value of 0, an x-intercept may be found by substituting the number 0 for y and solving the resulting equation for x . By the same reasoning, the *y-intercept* occurs when x is zero. The use of the intercepts is often a convenient way to sketch a graph of a linear equation.

Let us consider Property 1.4 again for a moment. We stated that not both *a* and *b* could be zero. What would happen for $a \neq 0$ and $b = 0$? For example, what is the graph of $2x + 0y = 4$. We could state that $\{(x, y) \mid 2x + 0y = 4\} = \{(x, y) \mid x = 2\}$. Then we would have ordered pairs with the first coordinate always 2 and any real number for y. This graph is a line parallel to the y-axis, crossing the x-axis at the point (2,0). If $a = 0$ and $b \neq 0$, then we should expect to obtain a line parallel to the x-axis. Why is this true? Take time now to reason through this situation.

The equation of the form $ax + by + c = 0$ can be written in other forms. We often wish to solve for one of the variables, usually y, in terms of the other variable and constants. Thus, if $b \neq 0$ we may proceed as follows.

$$ax + by + c = 0$$
$$by = -ax + -c$$
$$y = \frac{-a}{b}x + \frac{-c}{b}$$

Now since a, b, and c are constants we may let $m = \frac{-a}{b}$ and $k = \frac{-c}{b}$. Then we could write $y = mx + k$. This particular form of the linear equation is actually useful to us. Notice, for example, that if $x = 0$, then $y = k$ and thus the constant term k always gives us the y - intercept. Notice further that if (x_1, y_1) and (x_1, y_2) are two ordered pairs in the solution set, then it must be true that $y_1 = m x_1 + b$ and $y_2 = m x_2 + b$. Thus, $b = y_1 - m x_1$ and $b = y_2 - m x_2$, so $y_1 - m x_1 = y_2 - m x_2$. Then $m x_2 - m x_1 = y_2 - y_1$ so $m(x_2 - x_1) = y_2 - y_1$ and $m = \frac{y_2 - y_1}{x_2 - x_1}$. Now since m is a constant. this tells us that for any points (x_1, y_1) and (x_2, y_2) in the solution set, the difference in the y values divided by the difference in the x values is constant. This ratio, $\frac{y_2 - y_1}{x_2 - x_1}$, is called the slope of the line. We call the equation y = mx + b the slope-intercept form. For emphasis, let us further dignify the concept of slope by a definition.

Definition of Slope - By the slope of a line passing through distinct points P(x_1, y_1) and Q(x_2, y_2) we shall mean the real number m = $\frac{y_2 - y_1}{x_2 - x_1}$ if m exist.

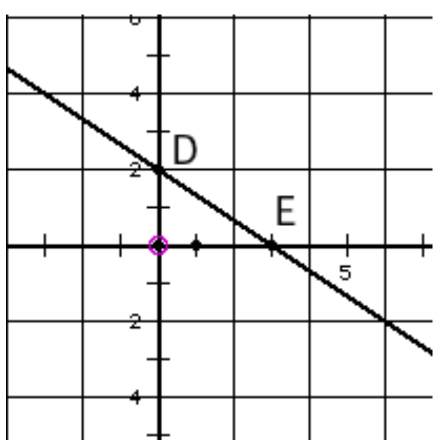
Example 1 A straight line passes through points P(2,3) and Q(3,5). What is the slope of the line PQ?

Solution m = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 5}{2 - 3} = \frac{-2}{-1} = 2$. Note also that m = $\frac{5 - 3}{3 - 2} = \frac{2}{1} = 2$ so we could choose either point to represent (x_1, y_1).

Example 2 Find the slope 2x + 3y = 6.

Solution a

Let us use the formula for slope m = $\frac{y_2 - y_1}{x_2 - x_1}$. We will use tow ordered pairs from the solution set. The points (0,2) and (3,0) will do. Them m = $\frac{2 - 0}{0 - 3} = \frac{2}{-3} = \frac{-2}{3}$. The graph of the line in the display below gives us some notion of a slope of $\frac{-2}{3}$.



Solution b

We could also have determined the slope by expressing the equation in y = mx + b form. The coefficient of x is the slope.

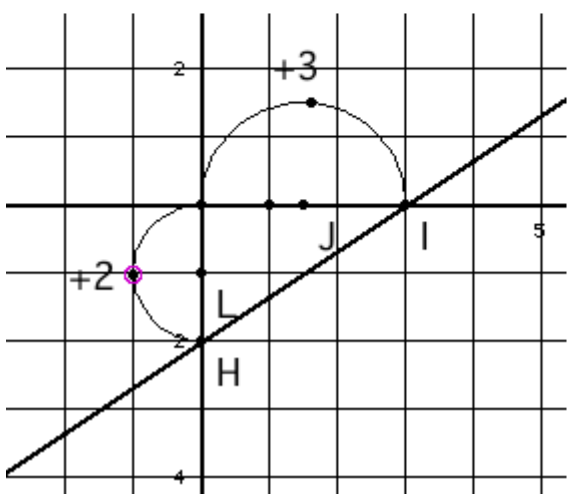
2x + 3y + 6

3y = -2x + 6

$y = \frac{-2}{3}x + 2$

therefore $m = \frac{-2}{3}$

The slope of a line is useful in many ways. Referring to Example 2, let us relate the slope of $\frac{2}{3}$ to the graph of 2x - 3y = 6 (or to the equivalent equation, $y = \frac{2}{3}x - 2$). In general a slope of $\frac{2}{3}$ means that starting at any point on the line, if we allow y to increase by 2 and then increase x by 3 we should have a point on the line again. Since we know that the y-intercept is -2, let us start at (0, -2) and change y by 2. Now change x by 3 and we are at the point (3,0). [Notice that this ordered pair (3,0) is in the solution set.] Thus, the graph must appear as in the display below. If an equation is expressed in slope intercept form, we may easily sketch a graph using this technique.



Suppose A(1,4), B(2,3), and C(5,0) are given as three ordered pairs belonging to the solution set of some linear equation. What is the slope of the line? The formula for slope requires the use on only two points. Since the points A, B, and C are collinear, and since the line has a given slope, we expect that slope not to be dependent upon the particular choice of points. Let us verify this fact for this example by working out slopes for all possible pairs of the given points.

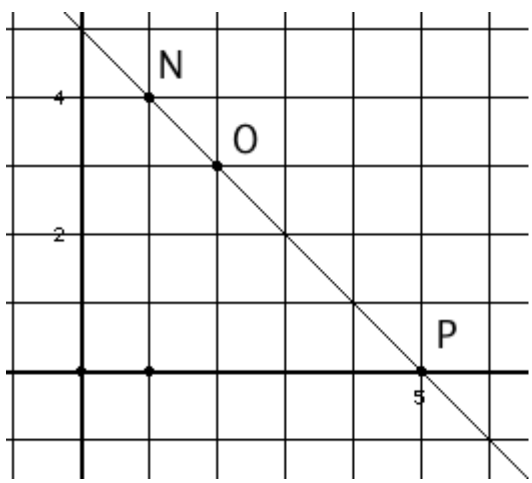
$$m(AB) = \frac{4 - 3}{1 - 2} = \frac{1}{-1} = -1 \left(\text{note also } \frac{3 - 4}{2 - 1} = \frac{-1}{1} = -1\right)$$

$$m(AC) = \frac{4 - 0}{1 - 5} = \frac{4}{-4} = -1 \ (\text{also } \frac{0 - 4}{5 - 1} = \frac{-4}{4} = -1)$$

$$m(BC) = \frac{0 - 3}{5 - 2} = \frac{-3}{3} = -1 \ (\text{also } \frac{3 - 0}{2 - 5} = \frac{3}{-3} = -1)$$

The slope of the line in question must be -1 and each representation of the line produced the same slope.

Suppose that we now wish to write the equation of the line graphed in the display below. We know that m = -1. If we could determine the y-intercept we could immediately write the desired equation. In this case, since the slope is -1 we could reason that the point one unit to left of (1,4) and one unit up is on the line. The y-intercept is thus (0,5) and the equation is y = -1x + 5. Now let us check our results. Notice that $4 = -1 \cdot 1 + 5$, that $3 = -1 \cdot 2 + 5$, and that $0 = -1 \cdot 5 + 5$. Therefore all three of the points check and the equation is correct.



Let us consider a more general approach to the problem just solved. Suppose that P(x, y) is any point different from the point having coordinates (2,3). The slope relative to these two points could be expressed as $m = \frac{y - 3}{x - 2} = -1$; thus y -3 =-1(x -2), and we may write y = -x + 5.

This procedure, using the general point P(x, y) as one of the points in the set of ordered pairs, is a very effective tool for expressing general relationships about points.

Example 1 Write the equation of the line determined by A(-3, 1) and B(2, 5).

Solution a Using points A and B, $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{2 - {}^{-}3} = \frac{4}{5}$. Using point P(x, y) different from A and B, we write $m = \frac{y - 5}{x - 2}$. Thus

$$\frac{y - 5}{x - 2} = \frac{4}{5}$$

$$y - 5 = \frac{4}{5}(x - 2)$$

$$y - 5 = \frac{4}{5}x - \frac{8}{5}$$

$$y = \frac{4}{5}x - \frac{8}{5} + 5$$

$$y = \frac{4}{5}x + \frac{17}{5}$$

check

$$1 \stackrel{?}{=} \frac{4}{5}({}^{-}3) + \frac{17}{5} \qquad \text{and} \qquad 5 \stackrel{?}{=} \frac{4}{5} \cdot 2 + \frac{17}{5}$$

$$1 \stackrel{?}{=} \frac{-12}{5} + \frac{17}{5} \qquad\qquad 5 \stackrel{?}{=} \frac{8}{5} + \frac{17}{5}$$

$$1 = \frac{5}{5} \qquad\qquad 5 = \frac{25}{5}$$

Classwork/Homework

Problems set 1 Using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, find the slope for each equation.

a) x + y = 3
b) 8 = x - y
c) x + y = 0
d) 2x = 6

Problem set 2 Find the intercepts for each of the following, then draw the graphs using the intercepts.
a) x + y = 5
b) 3x + 9 = y
c) 3x - 2y = 4

Problem set 3 Write each of the equations in slope-intercept form, then use the slope and y-intercept to produce the graph of the equation.
a) x + y = 7
b) -2x + y =0
c) 2y - x + 6 = 0
d) y + 3 = 0

Problem set 4 Find the slope of the line AB that passes through the given points.
a) A(4, 3), B(7, 5)
b) A(-3, -2), B(0,7)
c) A(-5, -8), B(-1, -32)
d) A($\frac{2}{3}$, $\frac{1}{2}$), B($\frac{3}{4}$, -2)

Problem set 5 Write the equation for each of the lines identified in Problem set 4.