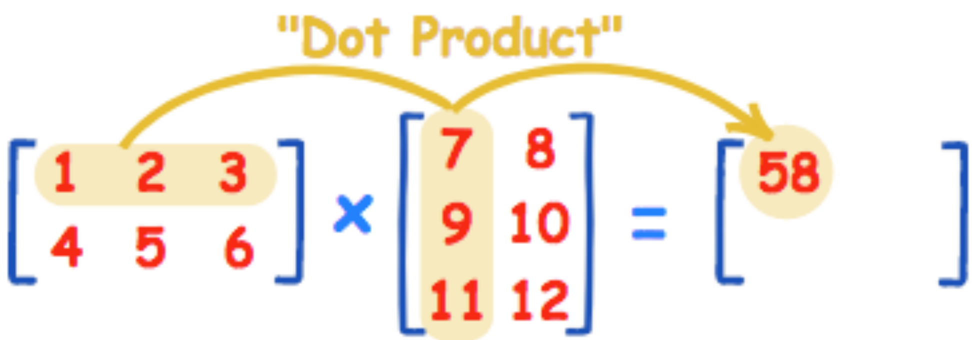
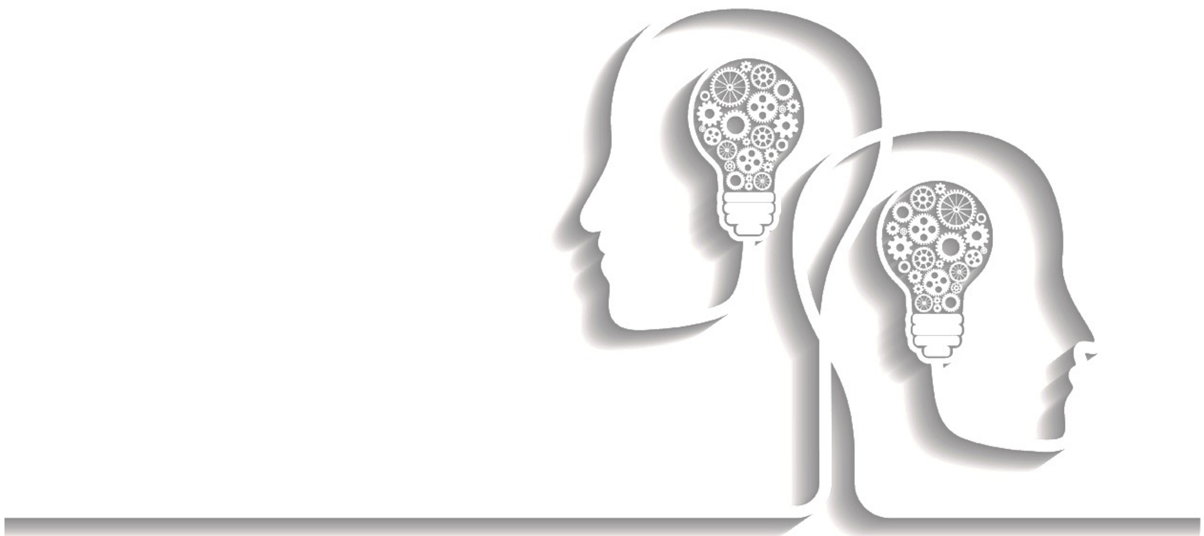
Multiplying Matrices

Booklet Three





Prepared by
Dr. Robert E. Mason IV
Mathematics Consultant

Multiplying Matrices

Objectives

After studying this booklet, you should be able to:

- Multiply matrices

Application: Sales

The manager of Fay's Donuts makes a daily report to the owner that summaries the cost of each kind of donut and the number of donuts sold for that day. The sales for one day are summarizes in the cost matrix *C* and sales matrix S shown below.

$$C = \begin{bmatrix} \textit{plain} & \textit{jelly} & \textit{glazed} & \textit{Specialty} \\ 0.45 & 0.55 & 0.50 & 0.85 \end{bmatrix} \quad S = \begin{bmatrix} & \textit{number} \\ \textit{plain} & 191 \\ \textit{jelly} & 122 \\ \textit{glazed} & 98 \\ \textit{specialty} & 69 \end{bmatrix}$$

You can use matrix multiplication to find the income for the day. In this case, multiply each element in the cost by its corresponding element in the sales matrix and find the total.

$$\cdot S = \begin{bmatrix} 0.45 & 0.55 & 0.50 & 0.85 \end{bmatrix} \cdot \begin{bmatrix} 191 \\ 122 \\ 98 \\ 69 \end{bmatrix} =$$

$$= \{0.45(191) + 0.55(122) + 0.50(98) + 0.85(69)\}$$
$$= \{85.95 + 67.10 + 49.00 + 58.65$$
$$= \{260.70\}$$

The income for the day was $260.70.

Multiplying Matrices	The product of A_{mxn} *and* B_{nxr} *is* $(AB)_{mxr}$. The element in the ith row and the jth column of AB is the sum of the products fo the corresponding elements in the ith row of A and the jth column of B.

Note the you can multiply two matrices only if the number of column in the first matrix is equal to the number of rows in the second matrix.

$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & -9 & 2 \\ 5 & 7 & 6 \end{bmatrix}$	$\begin{bmatrix} 3 & -9 & 2 \\ 5 & 7 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$
Possible because *2x2 2x3* *row x column*	*Not Possible* *2x3 2x2* *row x column*

The product on the left is defined, but the product on the right is not.

Example 1 – If A = $\begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix}$ and B = $\begin{matrix} 5 & 1 & -3 \\ 8 & -4 & 9 \end{matrix}$, find AB.

AB = $\begin{matrix} 3(5)+(-5)(8) & 3(1)+(-5)(-4) & 3(-3)+(-5)(9) \\ 2(5)+7(8) & 2(1)+7(-4) & 2(-3)+7(9) \end{matrix}$

= $\begin{matrix} 15-40 & 3+20 & -9-45 \\ 10+56 & 2-28 & -6+63 \end{matrix}$

= $\begin{matrix} -25 & 23 & -54 \\ 66 & -26 & 57 \end{matrix}$

Modeling: Rotations Matrices

We can use matrix multiplication in transformation geometry. Already learned how to translate a geometric figure and change its size by using matrices.

Another type of transformation is a **rotation**. A rotation occurs when a figure is moved around a center point. To move a figure by rotation, you can use a ***rotation matrix***.

The matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ will **rotate** a figure on the coordinate plane about the origin. In this activity, you will determine the direction and degrees of rotation. You will need grid paper, tracing paper, and a protractor to complete this activity.

What to do

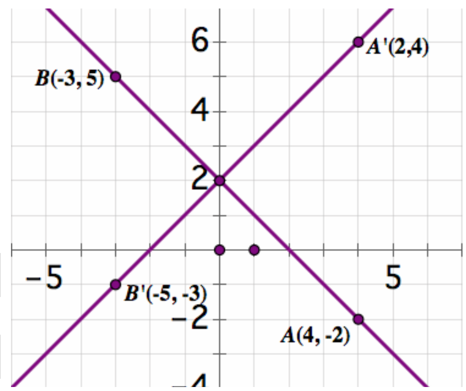
a. Draw a triangle on a coordinate plane and label it ΔABC. Write the coordinates of the vertices as a **coordinate matrix**.
b. Multiply the rotation matrix shown above by your **coordinate matrix**. Graph the resulting triangle on the same coordinate plane and label it ΔA'B'C'. Note that the rotation matrix should be on the left when multiplying.
c. Place a piece of tracing paper over ΔABC and trace it. With your pencil at the origin as a pivot point, slowly turn the tracing paper until the drawing of ΔABC matches ΔA'B'C'. Describe the motion of the triangle.
d. On the coordinate plane, draw $\overline{OA}$ and $\overline{OA'}$. Find the measure of ∠ A'O'A. Repeat for the remaining vertices.
e. Write a sentence that describes the effect of multiplying a **coordinate matrix** by **the rotation matrix** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Example 2 – Integration: Geometry

Line AB passes through points A(4, -2) and B(-3, 5). Find coordinates of two points on line A'B' that has been rotated 90^0 counterclockwise about the origin. Draw its graph and describe the relationship between lines AB and A'B'.

Write the ordered pairs in a coordinate matrix. Then multiply the coordinate matrix by the rotation matrix.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 4 & -3 \end{bmatrix}$$



Coordinates of two points on the line are A'(2, 4) and B'(-5, -3). The lines appear to be perpendicular.

Check for Understanding

Study the lesson. Then complete the following in your toolbox-book.

1. **Name** the conditions under which two matrices can be multiplied.
2. **Find** the dimensions of matrix M if M = A_{3x2} x B_{2x4}.
3. **Write** a convincing argument for the statement Matrix multiplication is commutative. If the statement is not true, find a counterexample.
4. **Give an example** of two matrices M and N for which the products MN and NM are both defined.
5. **You Decide** Brandon thinks two matrices can always be multiplied if they can be added. Dolores thinks that isn't necessarily true. Who is correct? Explain your reasoning.
6. Apply a 90-degree counterclockwise rotation about the origin twice to a triangle on the coordinate plane. Compare the new coordinates to the original ones. Make a conjecture about what effect this rotation has on any figure.

Guided Practice

Find the dimensions of each matrix product.

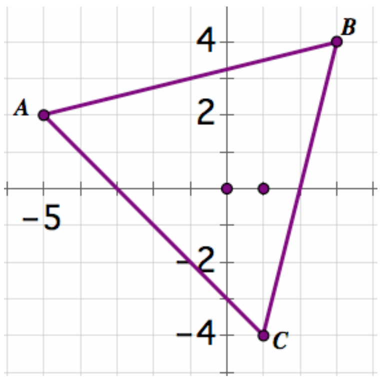
7. $A_{3x5} \cdot B_{5x2}$
8. $P_{2x2} \cdot Q_{2x4}$

Perform the indicated operations, if possible.

9. $\begin{bmatrix} 4 & -2 & -7 \\ 6 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$

10. $\begin{bmatrix} 4 & -1 & 6 \\ 1 & 5 & -8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 9 & -6 \end{bmatrix}$

11. Geometry Find the new coordinates of the vertices of triangle ABC with vertices A(-5, 2), B(3, 4), and C(1, -4). When the triangle is rotated 90^0 counterclockwise about the origin. Graph the original triangle and its rotation A'B'C'.



EXERCISES – Practice

Find the dimensions of each matrix product.

12. $A_{5x2} \times B_{2x5}$
13. $M_{4x2} \times N_{1x3}$
14. $R_{2x3} \times S_{3x4}$
15. $X_{3x4} \times Y_{4x1}$
16. $P_{1x4} \times Q_{5x1}$
17. $A_{3x2} \times B_{3x2}$

Perform the indicated operations, if possible.

18. $\begin{bmatrix} 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 3 \end{bmatrix}$	19. $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -9 & -2 \\ 5 & 7 & -6 \end{bmatrix}$
20. $\begin{bmatrix} 4 & -1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 4 \end{bmatrix}$	21. $\begin{bmatrix} 5 & -2 & -1 \\ 8 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$

Perform the indicated operations, if possible.

22. $3\begin{bmatrix} 5 & 7 \\ 1 & -2 \end{bmatrix} + 2\begin{bmatrix} -3 & 0 \\ -4 & 2 \end{bmatrix}$	23. $\begin{bmatrix} 0 & 8 \\ 3 & 1 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & -2 \\ 0 & 8 & -5 \end{bmatrix}$

24. **Geometry** Find the new coordinates of the vertices of $\triangle ABC$ with coordinates A(3, 5), B(6, 5), and C(0, 0) after it has been rotated 90^0 counterclockwise about the origin.

25. **Geometry** Given that A is any 2 x 2 matrix, and R is the rotation matrix, is RA = AR? Explain.

Use the matrices A, B, C, and D to evaluate each expression.

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 & -3 \\ 7 & -5 & 9 \end{bmatrix} \quad C = \begin{bmatrix} -6 & 4 \\ -2 & 8 \\ 3 & 0 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 \\ 3 & 7 \end{bmatrix}$$

26. AB + B 27. CB +B 28. AD + CB 29. AD + BC

30. **Geometry** After a triangle was rotated 90^0 counterclockwise about the origin, the coordinates of the vertices are (-3, -5), (-2, 7), and (1, 4). What were the coordinates of the vertices of the triangle in its original position?

Critical Thinking

31. Find the values of w, x, y, and z to make the statement $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ true. If the matrix containing w, x, y, and z were multiplied by any other matrix containing two columns, what do you think the result would be?

Test Your Understanding

(Self-test)

Multiplication Matrices

Find each matrix product when possible. Show all work. Total points 32.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 7 \end{bmatrix}$

2. $\begin{bmatrix} 3 & -4 & 1 \\ 5 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} \sqrt{3} & 1 \\ 2\sqrt{5} & \sqrt{27} \end{bmatrix}\begin{bmatrix} \sqrt{3} & -\sqrt{6} \\ 4\sqrt{3} & 0 \end{bmatrix}$

4. $\begin{bmatrix} -3 & 0 & 2 & 1 \\ 4 & 0 & 2 & 6 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 4 \\ 2 & 1 & 0 \\ 0 & -1 & 4 \end{bmatrix}$

6. $\begin{bmatrix} -2 & -3 & -4 \\ 2 & -1 & 0 \\ 4 & -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$

Key

1. $\begin{bmatrix} 13 \\ 25 \end{bmatrix}$ 4 points

2. $\begin{bmatrix} -17 \\ -1 \end{bmatrix}$ 4 points

3. $\begin{bmatrix} 3+4\sqrt{3} & -3\sqrt{2} \\ 2\sqrt{15}+36 & -2\sqrt{30} \end{bmatrix}$ 8 points

4. Cannot be multiplied

5. $\begin{bmatrix} 2 & 7 & -4 \end{bmatrix}$ 6 points

6. $\begin{bmatrix} -15 & -16 & 3 \\ -1 & 0 & 9 \\ 7 & 6 & 12 \end{bmatrix}$ 9 points