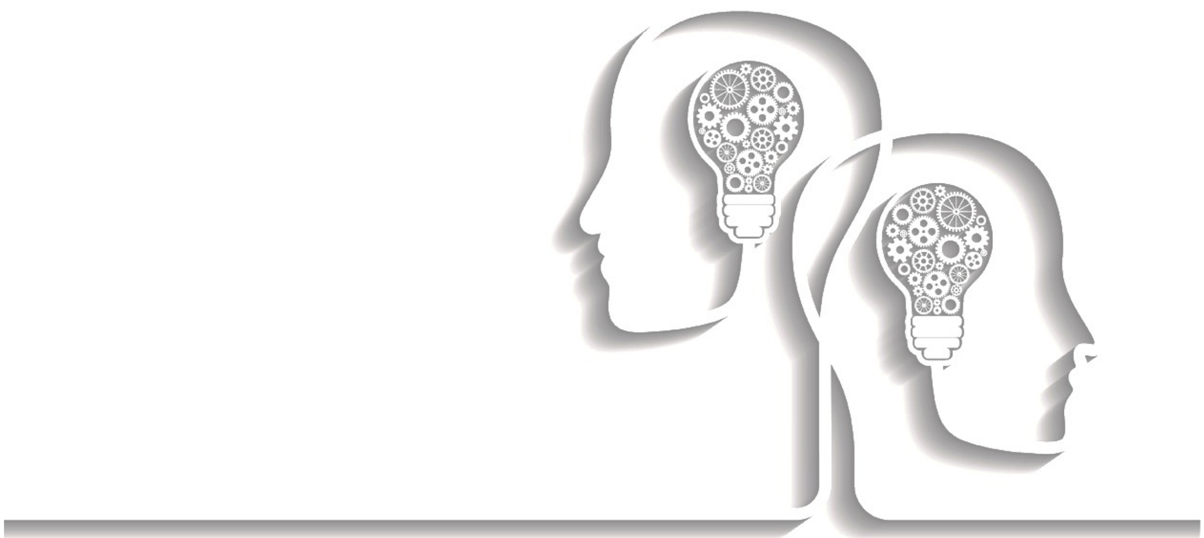
An Introduction to Matrices

Booklet One

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$



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Objectives

After studying this assignment, you should be able to:

- Perform scalar multiplication on a matrix,
- Solve matrices for variables, and
- Solve problems using matrix logic.

Application: Which College to Choose?

John has been accepted at three colleges in New York: Hunter College, Brooklyn College, and City College of New York (CCNY). He and his parents are trying to make a final decision based on cost, distance from home, campus life, and educational quality. John rates each criterion on a scale from 1 (least favorable) to 10 (most favorable) and organizes the information in a matrix like the one shown below. A matrix is a rectangular array of variables or constants in horizontal rows and vertical columns. Columns, usually enclosed in brackets. Matrices are often used as a problem-solving tool.

	Cost	Distance	Campus	Quality
Hunter College	5	10	3	10
Brooklyn College	5	5	5	8
City College	5	1	10	8

When the information is shown in a matrix, it is easy to see that cost is not useful criteria, because each college received the same score. You can also see that Hunter College is most favorable in terms of distance from John's home and the quality of education. Based on these criteria, John should attend Hunter College of the City University of New York.

In a matrix, numbers or data are organized so that each position on the matrix has a purpose. Each value in the matrix is called an element.

C =	5	10	3	10	3 Rows
	5	5	5	8	
	5	1	10	8	
	4 columns				

The element 10 is in row 1, column 2

A matrix is usually named using an uppercase letter, as in matrix *C* above. A matrix can also be named by using the matrix dimensions with the letter name. The matrix above would be names $C_{3 \times 4}$ since it has 3 rows and 4 columns.

Many problems can be solved using a method often called matrix logic. When you use matrix logic, you create a matrix that helps you organize all the information in the problem. By using the matrix, you can eliminate one possibility after another until you eventually arrive at a solution.

Example 1 - Problem Solving Using Matrix Logic

Fay, Amanda, Paola, and Miko are friends, and each has one of these pets: dog, cat, parrot, and gerbil. Use these clues to match each girl with her pet.

- Paola likes to visit the friend with the gerbil.
- Miko and Amanda frequently help their friend walk her dog.
- Fay cannot have a dog or a cat because she is allergic to them.
- Miko plans to teach her pet how to talk.

Explore	There are 4 girls and 4 pets. You must match each girl with her pet by using the information from the statements above.
Plan	Make a 4 x 4 matrix to organize the information. Through the process of elimination each girl can be matched with her pet.
Solve	Put an X in the first row under gerbil to show that Paola does not have the gerbil. Put two Xs to show that Miko and Amanda do not own the dog. Put two Xs to show that Fay cannot have the cat or dog. By the process of elimination, Paola owns the dog. Put a circle in this box. Since only one girl owns the dog, put Xs in the rest of the boxes in that row. Continue to eliminate possibilities in this manner

	Dog	Cat	Parrot	Gerbil
Fay	X	X	X	O
Amanda	X	O	X	X
Paola	O	X	X	X
Miko	X	X	O	X

Fay has the gerbil, Amanda has the cat, Paola has the dog, and Miko has the parrot.

Check the result against the statements. The first statement says that

Paola likes to visit the girl with the gerbil, and the answer says that Fay has the gerbil. There is not conflict here. Using the same method for each sentence, you can see that there are no conflicts.

Although matrices are sometimes used as a problem-solving tool, their importance extends to another branch of mathematics called discrete mathematics. Discrete mathematics deals with finite or discontinuous quantities. This distinction between continuous and discrete quantities is one that you can encountered before. Think of a staircase. You can slide your hand up the banister, but you have to climb the steps one by one. The banister represents a continuous quantity, like a linear function. However, each step represents a discrete quantity, like a point on a scatter plot or an element of a matrix.

Just as algebraic rules exist for functions, matrices have special algebraic rules. For example, you can multiply any matrix by a constant called a scalar. This is called scalar multiplication. When scalar multiplication is performed, each element is multiplied by the constant, and a new matrix is formed.

Scalar Multiplication of a Matrix

$$k\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$$

Example 2 – Application Business

The manager of Great Sports keeps track of monthly sales on a spreadsheet. The spreadsheet below shows the number of baseball and softball bats, ball, shoes, and gloves sold last May. This May, the store is going to have a promotion and hopes to increase sales by 8%. Write the that shows the store's goals for this May.

	A	B	C	D	E
1		bats	balls	shoes	gloves
2	baseball	38	29	18	43
3	softball	42	25	16	51

First, write the matrix for last May.

$$\begin{bmatrix} 38 & 29 & 18 & 43 \\ 42 & 25 & 16 & 51 \end{bmatrix}$$

Multiply the matrix by 1.08 to show an increase of 8% for this May.

$$1.08\begin{bmatrix} 38 & 29 & 18 & 43 \\ 42 & 25 & 16 & 51 \end{bmatrix} = \begin{bmatrix} 41.04 & 31.32 & 19.44 & 46.44 \\ 45.36 & 27.00 & 17.28 & 55.08 \end{bmatrix}$$

The Definition of Equal Matrices

Two matrices are considered to equal if they have the same dimensions and if each element of one matrix is equal to the corresponding element of the other matrix.
The definition of equal matrices can be used to find values when elements of the matrices are algebraic expressions.

Example 3 Solve $\begin{bmatrix} 2x \\ 2x+3y \end{bmatrix} = \begin{bmatrix} y \\ 12 \end{bmatrix}$ for x and y.

Since the matrices are equal, the corresponding elements are equal. When you write the sentences that show this equality, two linear equations are formed.

$$2x = y$$
$$2x + 3y = 12$$

The first equation gives you a value for y that can be substituted into the second equation. Then you can find a value for x.

$2x + 3y = 12$

$2x + 3(2x) = 12$ Replace y with 2x.

$2x + 6x = 12$ Simplify.

$8x = 12$ Combine like terms.

$x = 1.5$ Divide each side by 8.

To find a value for you can substitute 1.5 into either equation.

$2x = y$

$2(1.5) = y$ Replace x with 1.5.

3=y

The solution is (1.5, 3)

Check your solution by substituting the values into the equation you did not use to find y.

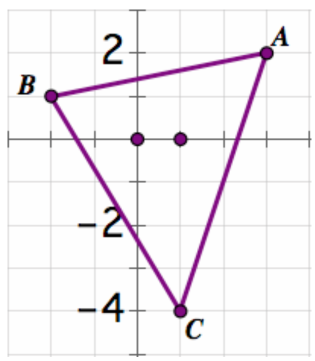
$2x + 3y = 12$

$2(1.5) + 3(3) = 12$ Replace x with 1.5 and y with 2.

12=12

The Coordinate Matrix

Matrices are an important tool for integrating algebra and geometry because points and polygons can be represented by matrices. The ordered pair (x, y) is usually represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, where the x-coordinate is in row 1, and the y-coordinate is in row 2. Similarly, polygons can be represented by grouping all of the columns matrices of the coordinates of the vertices into one matrix.



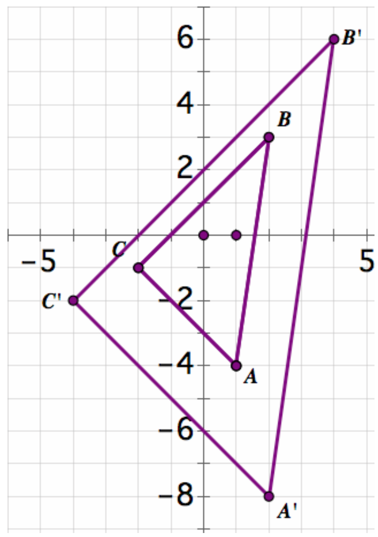
Coordinates of vertices

$$\Delta ABC = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -4 \end{bmatrix} \begin{matrix} \leftarrow \underline{\quad x\text{-coordinate} \quad} \\ \leftarrow \underline{\quad y\text{-coordinate} \quad} \end{matrix}$$

One of the ways that matrices help connect algebra and geometry is through **transformations**. **Transformations** are functions that map points of a shape onto its image. When a geometric figure is enlarged or reduced, this transformation is called a **dilation.** When the size of figures changes, all linear measures of it image change in the same ratio ***(this is why you studied fractions, ratios, and proportions in the elementary grades).*** For example, if the perimeter of a figure triples, the length of each side of the figure also triples.

Example 4 – Integrating Algebra and Geometry

Triangle ABC has vertices A(1,-4), B(2,3), and C(-2, -1). Enlarge triangle ABC so that its perimeter is twice the original perimeter. What are the coordinates of the vertices of triangle A'B'C'?



Graph triangle ABC. Since the perimeter is linear measurement, multiply the coordinate matrix, by the scalar 2.

$$2\begin{bmatrix} 1 & 2 & -2 \\ -4 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -4 \\ -8 & 6 & -2 \end{bmatrix}$$

The coordinates of the vertices of triangle A'B'C' are A'(2,8), B'(4,6), and C'(-4,-2). Graph triangle A'B'C'.

You should measure to verify that the perimeter of triangle A'B'C' is twice the original perimeter.

Check for Understanding

Study the lesson. Then complete the following in your toolbox-book.

1. **Define** a matrix in your own words.
2. **Find** an example of a matrix in a newspaper and name it using its dimensions.
3. **Choose** the matrix that represents the ordered pair (-1, 3).

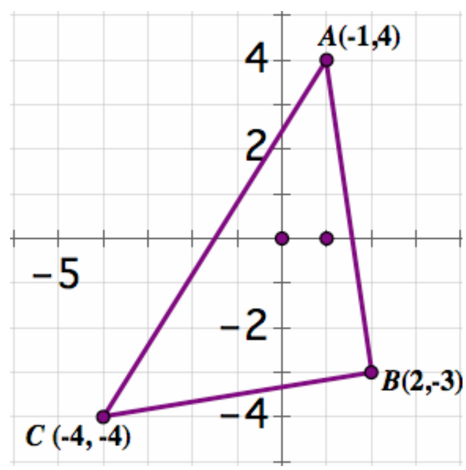
a. [-1,3]

b. $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

c. [3, -1]

d. $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

4. **Write** a coordinate matrix for the triangle shown at the right.
5. **Explain** the meaning of *dilation.*



Guided Practice

Perform the indicted operations

7. $-2\begin{bmatrix} 7 & 3 & -1 \end{bmatrix}$

8. $4\begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}$

Solve for the variables.

9. $\begin{bmatrix} 2x & 3 & 3z \end{bmatrix} = \begin{bmatrix} 5 & 3y & 9 \end{bmatrix}$

10. $\begin{bmatrix} 6x \\ y \end{bmatrix} = \begin{bmatrix} 62+8y \\ 6-2x \end{bmatrix}$

11. Business - On Monday, the Main Street Deli sold the following number of sandwiches: 15 turkeys, 12 turkey and cheese,8 ham, 10 ham and cheese, 8 roast beef, 11 roast beef and cheese. Organize the information into a 3 x 2 matrix.

12. Geometry – Triangle ABC with A(4,5), B(-3,-2), and C(1, -4) is reduced so that its perimeter is one-half the original perimeter.
a. Write the coordinate matrix for triangle ABC.
b. Write the coordinates of A'B'C' in matrix form.
c. Graph this situation.

EXERCISES - Practice

Perform the indicated operation.

13. $3\begin{bmatrix} 5 & -2 & 7 \\ -3 & 8 & 4 \end{bmatrix}$

14. $-2\begin{bmatrix} -6 & -4 \\ -2 & 4 \end{bmatrix}$

15. $\frac{1}{3}\begin{bmatrix} 6 & -5 \end{bmatrix}$

16. $0.2\begin{bmatrix} 10.50 \\ 8.75 \end{bmatrix}$

17. $-5\begin{bmatrix} 1.3 & 0 & 5.1 \\ 0.4 & 1.0 & 2.5 \end{bmatrix}$

18. $-0.3\begin{bmatrix} 8.95 & 7.50 \end{bmatrix}$

Solve for the variables.

19. $\begin{bmatrix} 4x & 3y \end{bmatrix} = \begin{bmatrix} 12 & -1 \end{bmatrix}$

20. $\begin{bmatrix} 2x+y \\ x-2y \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$

21. $x\begin{bmatrix} 4 & y \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 12 & -15 \\ 21 & z \end{bmatrix}$

22. $4\begin{bmatrix} x & y-1 \\ 3 & z \end{bmatrix} = \begin{bmatrix} 20 & 8 \\ 6z & x+y \end{bmatrix}$

23. $\begin{bmatrix} x^2 & 7 & 9 \\ 5 & 12 & 6 \end{bmatrix} = \begin{bmatrix} 25 & 7 & y \\ 5 & 2z & 6 \end{bmatrix}$

24. $\begin{bmatrix} x+3y \\ 3x+y \end{bmatrix} = \begin{bmatrix} -13 \\ 1 \end{bmatrix}$

25. Geometry - The vertex of the right angle of a right triangle is located at the origin with its other vertices at (0,12) and (5,0). Find the coordinates of the vertices of a similar triangle whose perimeter is one-fourth that of the original triangle.

26. Geometry – The coordinate matrix for triangle XYZ is $\begin{bmatrix} -2 & 4 & -1 \\ -1 & 2 & 3 \end{bmatrix}$. Explain what happens to the triangle when the matrix is multiplied by -0.5. Make a drawing of justify your answer.

Solve for the variables.

27. $\begin{bmatrix} r^2-24 & 17 \\ 7 & t^3 \end{bmatrix} = \begin{bmatrix} 1 & 2y+3 \\ z^2-12 & 27 \end{bmatrix}$

28. $\begin{bmatrix} 5x-7 & 11 \\ 5 & 23 \end{bmatrix} = \begin{bmatrix} 8 & 21-m \\ r-3 & 4y+x \end{bmatrix}$

29. $\begin{bmatrix} 13-7y & a \\ 1 & 2b-38 \end{bmatrix} = \begin{bmatrix} 5x & 2-6b \\ 2x+3y & 5a \end{bmatrix}$

Dare To Think

31. When the size of a figure changes, all linear measures of its image, such as the perimeter, change in the same ratio. Is it also true that the area of the figure changes in the same ratio? Justify your answer with matrices and a graph.

Applications and Problem Solving

32. **Use Matrix Logic** – Fred, Ted, and Ed are going with Mary, Carrie, and Terri to the math tournament as their partner. Use these clues to find which couples will be attending the tournament as partners.

- Mary is Ed's sister and lives on Fifth Avenue.
- Ted drives a car to school each day.
- Ed is taller than Terri's partner.
- Carrie and her partner rider their bicycles to school every day.
- Fred's partner lives on Wall Street.

Test Your Understanding

(Self-test)

Find the values of the variable for which each statement is true, if possible. Circle your answers. Show all work. Answers are provided at the end.

1. $\begin{bmatrix} w & x \\ 8 & -12 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ y & z \end{bmatrix}$

2. $\begin{bmatrix} 6 & a+3 \\ b+2 & 9 \end{bmatrix} = \begin{bmatrix} c-3 & 4 \\ -2 & d-4 \end{bmatrix}$

3. $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$

4. $\begin{bmatrix} 5 & x-4 & 9 \\ 2 & -3 & 8 \\ 6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} y+3 & 2 & 9 \\ z+4 & -3 & 8 \\ 6 & 0 & w \end{bmatrix}$

5. $\begin{bmatrix} a+2 & 3z+1 & 5m \\ 8k & 0 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2z & 5m \\ 2k & 5 & 6 \end{bmatrix} = \begin{bmatrix} 10 & -14 & 80 \\ 10 & 5 & 9 \end{bmatrix}$

6. $\begin{bmatrix} 2 & \sqrt{7} \\ 3\sqrt{28} & -6 \end{bmatrix} - \begin{bmatrix} -1 & 5\sqrt{7} \\ 2\sqrt{7} & 2 \end{bmatrix}$

7. $\begin{bmatrix} 4k-8y \\ 6z-3x \\ 2k+5a \\ -4m+2n \end{bmatrix} - \begin{bmatrix} 5k+6y \\ 2z+5x \\ 4k+6a \\ 4m-2n \end{bmatrix}$

Key

1. w=9 x=17 y=8 z=-12

2. a= 1 b=-4 c=9 d=13

3. Not Equal

4. x= 6 y=8 z=-2 w=5

5.

$$\begin{bmatrix} 4a+2=10 & 5z+1=-14 & 10m=80 \\ 10k=10 & 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} a=2 & z=-3 & m=8 \\ k=1 & 5 & 9 \end{bmatrix}$$

6.

$$\begin{bmatrix} 3 & -4\sqrt{7} \\ 4\sqrt{7} & -8 \end{bmatrix}$$

7.

$$\begin{bmatrix} -k-14y \\ 4z-8x \\ -2k-a \\ -8m+4n \end{bmatrix}$$